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Mathematical considerations regarding the toroidal momentum operator

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Abstract. After a short presentation of the toroidal moments and the necessity to introduce them in the multipole expansion of current density, the correspondent quantum operators are introduced. The toroidal momentum operator (the quantum operator corresponding to the lowest-order toroidal multipole) is analysed. A natural set of coordinates is found. Using this set of coordinates it becomes possible to find the eigenvalues and a complete orthonormal set of eigenfunctions of the projection of this operator on the Oz axis.

1. Introduction

Although the multipole decomposition of charge and current densities is almost as old as classical electrodynamics, a whole class of terms has remained unknown for a long time. The history of toroidal moments began with Zeldovich's pioneering work [1]. He was the first to note that a closed toroidal current (which cannot be reduced to a usual charge or magnetic multipole moment) represents a certain new kind of dipole.

After the discovery of parity violation in weak interactions, he considered a new kind of electromagnetic interaction (invariant under time reversal, but odd under parity) of the form

$$\mathcal{H} \sim \boldsymbol{S}\boldsymbol{J}^{\mathrm{ext}} = \boldsymbol{S}(\mathrm{rot}\,\boldsymbol{H}^{\mathrm{ext}})$$

where $\mathcal{H}$ is the Hamiltonian, $\boldsymbol{S}$ the spin operator, while $\boldsymbol{J}^{\mathrm{ext}}$ and $\boldsymbol{H}^{\mathrm{ext}}$ represent the external current and magnetic field. He observed that, if one allows for violations of the discrete spacetime symmetries, a spin-$\frac{1}{2}$ particle might possess, besides the usual electric and magnetic dipole moments, a third kind of dipole characteristic, which was named 'anapole' to distinguish it from the usual electric and magnetic dipoles.

In the work by Yu M Shirokov, A A Tcheshkov, and V M Dubovik (summarized in the reviews [2, 3]) it has been shown that there is a whole independent class of ('toroidal') multipole moments and Zeldovich's 'anapole' is a combination of the first term of this class and magnetic moments. A complete parametrization for the most general configuration of charges and currents has been obtained in terms of three families of electric, magnetic and toroidal multipole moments and distributions, generated by the three independent scalar functions η, ψ and χ existing in the problem. Mean radii of various orders can also be unambiguously defined for any fixed multipolarity order (2^l pole order, $l = 1, 2, \ldots$) and type (electric, magnetic and toroidal) which, together with the multipole moments, achieve

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a full characterization of an arbitrary kind of source. Moreover, the toroidal moments represent measurable quantities, as we shall see in the next section.

However, the importance of the toroidal moments is not just in the description of the current sources. They also represent measurable quantities with specific modes of interaction with the electromagnetic field. For example, as Dubovik and Tugushev wrote in [3], the toroidal dipole moment is the only characteristic having an interaction energy from which one can directly measure the displacement current. From this principle a new type of electromagnetic motor could also be constructed (see [3]).

In the next section, after a brief discussion concerning the multipole decomposition of the charge and current densities, we associate quantum operators to the multipole moments. Although this procedure is general, we apply it only for the toroidal dipole momentum (this operator has already been used, for example, in [6]). The necessity of studying this operator further is obvious, considering its large applicability in physics at any scale (subnuclear [4, 5], nuclear [4], atomic [6], molecular and condensed matter physics [3]). However, the most exciting fact for us is that there is a whole class of particles (the Majorana fermions and self-conjugate bosons [4, 5]) which are not allowed to possess any electromagnetic structure other than toroidal multipole moments. This comes from CPT invariance alone [7].

Meanwhile, the most important facts we need to know about an operator are its spectrum and its eigenfunctions. Consequently, we shall solve this problem for the case of the toroidal momentum operator in section 3.

2. Toroidal momentum in classical electrodynamics and quantum mechanics

It is not our aim here to discuss the expansion in multipole moments in detail. We advise interested readers to see the very good paper of Dubovik and Tugushev [3]. Here we just state that (see also [4, 6]) in the multipole expansion of the current density, beginning with second order, apart from the usual electric and magnetic multipoles, the toroidal moments appear:

$$\begin{aligned} j_i(\boldsymbol{x}, t) = {}& \dot{Q}_i \delta^3(\boldsymbol{x}) - \left[\dot{Q}_{ik} - c\epsilon_{ikl} M_l + \frac{\delta_{ik}}{6}\dot{\bar{r}}^2\right] \partial_k \delta^3(\boldsymbol{x}) \\ & + \frac{1}{2}\left[2\dot{Q}_{ijk} + \frac{1}{10}(\delta_{ij}\dot{\bar{r}}_k^2 + \delta_{ik}\dot{\bar{r}}_j^2) - \frac{c}{3}(\epsilon_{ijl}M_{kl} + \epsilon_{ikl}M_{jl}) \right. \\ & \left. + c(\delta_{ik}T_j + \delta_{ij}T_k - 2\delta_{jk}T_i)\right] \partial_j \partial_k \delta^3(\boldsymbol{x}) + \cdots . \end{aligned} \tag{1}$$

In the expression above, Q and M represent electric and magnetic quantities and r denotes various mean radii (or time derivatives of these when overdotted). Apart from these usual terms new quantities appear: toroidal moments. In formula (1) we have the lowest-order terms from the whole class of these moments:

$$T_i = \frac{1}{10c} \int [\xi_i(\boldsymbol{\xi j}) - 2\boldsymbol{\xi}^2 j_i]\, \mathrm{d}^3\xi . \tag{2}$$

These terms could be interpreted as the projections of a vector $\boldsymbol{T}$ on the Cartesian axis. A straightforward interpretation of this vector is obtained if we expand the electromagnetic interaction energy:

$$H = \int \left(\rho\varphi - \frac{1}{c}\boldsymbol{jA}\right) \mathrm{d}^3x .$$

The interaction term corresponding to the toroidal momentum is [3, 4]

$$\begin{aligned} H(t)_{\text{tor}} &= -\boldsymbol{T}(t)[\operatorname{rot}\operatorname{rot}\boldsymbol{A}^{\text{ext}}(\boldsymbol{x},t)]_{x=0} \\ &= -\boldsymbol{T}(t)[\operatorname{rot}\boldsymbol{H}^{\text{ext}}(\boldsymbol{x},t)]_{x=0} \\ &= -\boldsymbol{T}(t)\left[\frac{4\pi}{c}\boldsymbol{J}^{\text{ext}}(\boldsymbol{x},t)+\frac{1}{c}\dot{\boldsymbol{D}}^{\text{ext}}(\boldsymbol{x},t)\right]_{x=0}. \end{aligned}$$

As we can see, the toroidal momentum interacts with the external conduction ($\boldsymbol{J}^{\text{ext}}$) and with the external displacement ($\dot{\boldsymbol{D}}^{\text{ext}}$) currents. The strength of the interaction is measured by the force momentum [4]:

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{T}\times(\operatorname{rot}\boldsymbol{H}^{\text{ext}}) \\ &= \boldsymbol{T}\times\left(\frac{4\pi}{c}\boldsymbol{J}^{\text{ext}}+\frac{1}{c}\dot{\boldsymbol{E}}^{\text{ext}}\right). \end{aligned}$$

The toroidal moments interact with the external field only if it overlaps with the source of the latter.

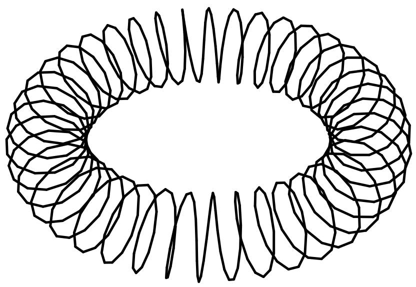

Figure 1. Toroidal solenoid.

As an example let us compute the toroidal momentum of a toroidal solenoid of large and small radii R_{T} and r_{T}, and with N turns of winding (see figure 1). If the symmetry axis of the toroidal solenoid lies along the z axis, then the projections of the toroidal and magnetic moments are (considering I the linear current in the wire)

$$\begin{aligned} &T_1 = T_2 = 0 && M_1 = M_2 = 0 \\ &T_3 = IN\left(\frac{\pi r_{\text{T}}^2 R_{\text{T}}}{10c}\right) = IN\frac{V_{\text{T}}}{5\pi c} && M_3 = I\pi(r_{\text{T}}^2+2R_{\text{T}}^2). \end{aligned}$$

If we replace the toroidal solenoid with N closed rings displayed in the same toroidal form, the projections become

$$\begin{aligned} &T_1 = T_2 = 0 \\ &T_3 = IN\frac{\pi r_{\text{T}}^2 R_{\text{T}}}{10c} = IN\frac{V_{\text{T}}}{5\pi c} \qquad M_1 = M_2 = M_3 = 0. \end{aligned}$$

Equation (2) is identically verified by the singular current density

$$\boldsymbol{j}_{\text{T}}(\boldsymbol{\xi}) = c\operatorname{rot}\operatorname{rot}\boldsymbol{T}\delta^3(\boldsymbol{\xi})$$

(with $\boldsymbol{T}$ a constant vector), which can be viewed as the current density of an elementary, point-like toroidal dipole [3, 4].

The above intuitive calculations lead us to the conclusion that the toroidal momentum can be viewed as a coil of magnetic field (in analogy with the magnetic moment, which is generated by a coil of current) and interacts with the sources of the electromagnetic field instead of the field itself. We now show further that this quantity also has an analogue in quantum mechanics.

Let us denote by $\Psi(\boldsymbol{x}, t)$ the wavefunction describing the state of a quantum particle. The current density corresponding to it (in the absence of an external electromagnetic field) has the form

$$\boldsymbol{j}(\boldsymbol{x}, t) = \frac{\mathrm{i}\hbar}{2m}(\Psi(\boldsymbol{x}, t)\boldsymbol{\nabla}\Psi^*(\boldsymbol{x}, t) - \Psi^*(\boldsymbol{x}, t)\boldsymbol{\nabla}\Psi(\boldsymbol{x}, t)).$$

As in the classical case, we can perform a multipole decomposition of the current density $\boldsymbol{j}$, and the toroidal momentum, computed according to (2), is

$$\begin{aligned} T_i &= \frac{1}{10c}\int [\xi_i(\boldsymbol{\xi}\boldsymbol{j}) - 2\boldsymbol{\xi}^2 j_i]\,\mathrm{d}^3\xi \\ &= \frac{1}{10mc}\int \sum_{k=1,2,3}\left\{(\xi_i\xi_k - 2\boldsymbol{\xi}^2\delta_{ik})\left[\frac{\mathrm{i}\hbar}{2}\left(\Psi(\boldsymbol{\xi}, t)\frac{\partial}{\partial\xi_k}\Psi^*(\boldsymbol{\xi}, t)\right.\right.\right. \\ &\qquad \left.\left.\left. -\Psi^*(\boldsymbol{\xi}, t)\frac{\partial}{\partial\xi_k}\Psi(\boldsymbol{\xi}, t)\right)\right]\right\}\mathrm{d}^3\xi. \end{aligned}$$

Because of the hermiticity of the operators $\hat{T}_i$ (defined as in [6])

$$\hat{T}_i = \frac{1}{10mc}\sum_{k=1,2,3}(x_i x_k - 2\boldsymbol{x}^2\delta_{ik})\hat{P}_k \qquad (3)$$

which we interpret as the projections of the ‘toroidal momentum operator’, the matrix elements of the toroidal momentum between any states can be written as

$$[T_i]_{mn} = \int \Psi_n^*(\boldsymbol{\xi}, t)\hat{T}_i(\boldsymbol{\xi})\Psi_m(\boldsymbol{\xi}, t)\,\mathrm{d}^3\xi$$

or, in Dirac notation,

$$[T_i]_{mn} = \langle n|\hat{T}_i|m\rangle.$$

So we can associate with the classical toroidal momentum (corresponding to a current density) a quantum operator, called the toroidal momentum operator. The physical significance of this operator is straightforward. It should be noted that in expression (3) the electromagnetic coupling constant e does not appear (as it does in [6]). We defined the operator in this way to emphasize that this is a characteristic of the particle state and not just a new kind of electromagnetic interaction.

3. The eigenvalues and eigenfunctions of the toroidal momentum operator

First we have to see if the projections of the toroidal momentum operator commute with each other. A straightforward calculation leads us to the result:

$$[\hat{T}_i, \hat{T}_j] = -\frac{6\mathrm{i}\hbar}{(10mc)^2}\boldsymbol{x}^2\epsilon_{ijk}\hat{L}_k.$$

A consequence of the above expression is that we cannot find a set of eigenfunctions common to two of the projections in the same time. We therefore have to solve the problem for just one of them. We choose $\hat{T}_3$.

We can also verify that $\hat{T}^2$ and $\hat{T}_i$ are not independent variables:

$$[\hat{T}^2, \hat{T}_i] = -\frac{32\mathrm{i}\hbar}{(10mc)^2}\boldsymbol{x}^2[\mathrm{i}\hbar\hat{T}_i - \epsilon_{ijk}\hat{L}_j\hat{T}_k].$$

Fortunately from the commutator

$$[\hat{T}_i, \hat{L}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{T}_k \tag{4}$$

we can see that $\hat{T}_3$ commutes with $\hat{L}_3$, so we can try to find a common set of eigenfunctions for these two operators.

3.1. A 'natural' set of coordinates for the toroidal momentum operator

Considering the commutator rules (4), we expect that $\hat{T}_3$ has a simpler form in cylindrical or spherical coordinates. This is indeed true and from now on we shall work in cylindrical coordinates:

$$\hat{T}_3 = \frac{-\mathrm{i}\hbar}{10mc}\left[\rho z\frac{\partial}{\partial\rho} - (2\rho^2 + z^2)\frac{\partial}{\partial z}\right].$$

We can see that the variable φ does not occur in the expression for $\hat{T}_3$.

In order to solve the problem we try to find a set of coordinates in which $\hat{T}_3$ has a very simple form. One of the possibilities is

$$\begin{aligned}\hat{T}_3 &= \frac{-\mathrm{i}\hbar}{10mc}\left[z\rho\frac{\partial}{\partial\rho} - (2\rho^2 + z^2)\frac{\partial}{\partial z}\right]\\ &= \frac{-\mathrm{i}\hbar}{10mc}\frac{\partial}{\partial u}.\end{aligned} \tag{5}$$

If we note the new set of variables by (k, u, φ), then, from equation (5), we find the following system of differential equations,

$$\begin{aligned}\frac{\partial\rho}{\partial u} &= \rho(k, u, \varphi)z(k, u, \varphi)\\ \frac{\partial z}{\partial u} &= -2\rho^2(k, u, \varphi) - z^2(k, u, \varphi)\end{aligned} \tag{6}$$

and the corresponding symmetric system:

$$\frac{\mathrm{d}\rho}{\rho z} = -\frac{\mathrm{d}z}{2\rho^2 + z^2} = \mathrm{d}u.$$

From the last system the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}\rho} = -\frac{2\rho^2 + z^2}{\rho z}$$

follows immediately, with two solutions:

$$\begin{aligned}z_+(\rho) &= \frac{\sqrt{-\rho^4 + C}}{\rho}\\ z_-(\rho) &= -\frac{\sqrt{-\rho^4 + C}}{\rho}.\end{aligned} \tag{7}$$

From the above equations we can find ρ as a function of z (in this case the solution is unique, considering that $\rho \geqslant 0$):

$$\rho = \sqrt{\frac{-z^2 + \sqrt{z^4 + 4C}}{2}}. \tag{8}$$

Derivatives along the curves defined by equations (7) and (8) (see figure 2) correspond to derivatives with respect to u (as u was defined in (5)) in the space of the new variables, (k, u, φ). Therefore along these curves k and φ are constants, and u varies in some range $(u_{\min}, u_{\max})$ which also depends on k and φ.

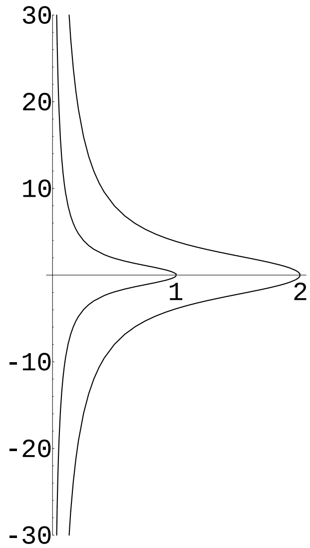

Figure 2. Curves corresponding to the new variable u, for $C = 1$ and $C = 16$ (or $k = 1$ and $k = 2$).

A simple inspection of equations (7) and (8) suggests that we put the variable k in correspondence with the integration constant C. For simplicity we choose

$$k = C^{1/4}$$

and from equations (7) or (8) we find

$$k = (\rho^2 z^2 + \rho^4)^{1/4}. \tag{9}$$

Now we can find u as a function of k and ρ, or k and z. To do this, we make use of equations (7) and (8) and we again write system (6):

$$\begin{aligned} \frac{\partial \rho}{\partial u} &= \pm\sqrt{-\rho^4 + k^4} \\ \frac{\partial z}{\partial u} &= -\sqrt{z^4 + 4k^4}. \end{aligned} \tag{10}$$

Once k is fixed, u can be determined as a function of ρ or z, but it will depend on an arbitrary constant. Let us choose this constant in such a way that $u(k, z)|_{z=0} = 0$ and $u(k, \rho)|_{\rho=k} = 0$ (k represents the maximum value of ρ on the curve $\rho = \rho(z, k)$, k being kept constant, as we can see from equations (7), (8) or (9)). Considering that we are working in a $\varphi =$ constant plane, we shall not mention this variable any longer to simplify the expressions. Using the above conventions we can write the solutions of system (10) as

$$u(k, z) = -\int_0^z \frac{1}{\sqrt{t^4 + 4k^4}}\, \mathrm{d}t \tag{11}$$

$$u(k,\rho) = \pm \int_\rho^k \frac{1}{\sqrt{-t^4+k^4}}\,\mathrm{d}t. \tag{12}$$

We now try to write the solutions above in a simpler form. We proceed with (12):

$$u(k,\rho) = \int_\rho^k \frac{\mathrm{d}t}{\sqrt{k^4-t^4}} = \int_0^k \frac{\mathrm{d}t}{\sqrt{k^4-t^4}} - \int_0^\rho \frac{\mathrm{d}t}{\sqrt{k^4-t^4}}.$$

After the substitution $t \to f(x) = kx^{1/4}$, the last integral on the right-hand side of the above expression becomes

$$\int_0^\rho \frac{\mathrm{d}t}{\sqrt{k^4-t^4}} = \frac{1}{4k}\int_0^{(\rho/k)^4} (1-x)^{-1/2}x^{-3/4}\,\mathrm{d}x = \frac{1}{4k}B_{(\rho/k)^4}\left(\frac{1}{4},\frac{1}{2}\right)$$

where $B_x(p,q)$ is the incomplete Beta function:

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,\mathrm{d}t$$

$$B_1(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

With this notation, u can be written as

$$\begin{aligned} u(k,\rho) &= \frac{1}{4k}\left[B_1\left(\frac{1}{4},\frac{1}{2}\right) - B_{(\rho/k)^4}\left(\frac{1}{4},\frac{1}{2}\right)\right] \\ &= \frac{1}{4k}B_1\left(\frac{1}{4},\frac{1}{2}\right)\left[1 - I_{(\rho/k)^4}\left(\frac{1}{4},\frac{1}{2}\right)\right] \end{aligned} \tag{13}$$

with the evident notation

$$I_x(p,q) = \frac{B_x(p,q)}{B_1(p,q)}.$$

We can also obtain the expression for u, when $u < 0$, by changing the sign on the right-hand side of formula (13). If in (13) we replace k with expression (9), we find u as a function of ρ and z.

Another expression for u can be obtained from equation (11):

$$u(k,z) = \pm\frac{(-1)^{1/4}}{4\sqrt{2}k}\int_0^{z^4/(4k^4)} (1-x)^{-1/2}x^{-3/4}\,\mathrm{d}x$$

where the '$+$' sign is for $z \leqslant 0$ and the '$-$' sign is for $z > 0$. We shall not use this formula further.

From the analysis of equation (13) we observe that u takes values in a finite interval which depends on k,

$$u \in (-a(k), a(k))$$

where

$$\begin{aligned} a(k) &\equiv \frac{C_\mathrm{a}}{k} \\ &= \frac{1}{4k}B_1\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4k}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} \sim \frac{1.311\,03}{k}. \end{aligned}$$

We can see that $a(k)$ behaves like $1/k$ (figure 3). For this reason the eigenfunctions of the toroidal momentum operator, which will form a complete set of orthogonal functions, will include a δ Dirac function of k.

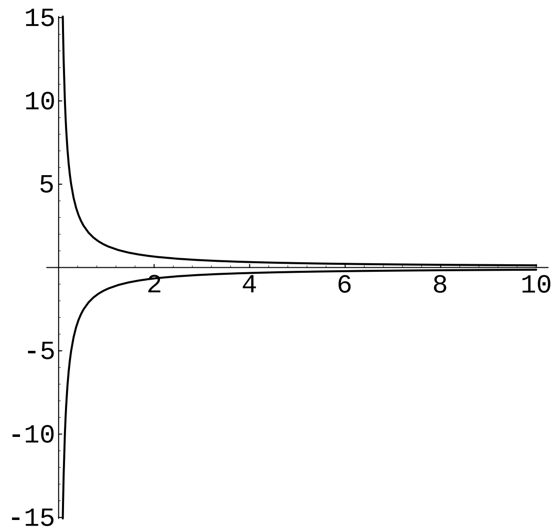

Figure 3. The limits of the intervals in which u takes values, at fixed k.

The last problem we are left with, in order to define the transformation completely, is to calculate the expression of the elementary volume $\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$ in the new coordinates:

$$\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \left\| \begin{matrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{matrix} \right\| \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3.$$

This can be done easily in two (obvious) steps:

$$\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = I_1\,\mathrm{d}\rho\,\mathrm{d}z\,\mathrm{d}\varphi = I_1 I_2\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}\varphi.$$

It is well known that $I_1 = \rho$, so we are just left with finding I_2. A straightforward, but tedious calculation, in which we make use of the equations (11), (12) and (10), gives us the result:

$$I_2 = \frac{2k^3}{\rho}.$$

For the total transformation, from the set of variables (x_1, x_2, x_3) to (k, u, φ), we obtain the very simple expression

$$I = I_1 I_2 = 2k^3.$$

This function could be interpreted as a weight for the scalar product in the space of the new variables, introduced in order to preserve the scalar product.

So we have defined a transformation from the variables (x_1, x_2, x_3) to the variables (k, u, φ). The functions in the (k, u, φ) variables are defined in a domain $\mathcal{D} = \{(k, u, \varphi)|k \in [0, \infty),\ \varphi \in [0, 2\pi),\ u \in (-a(k), a(k))\}$ (see figure 3) and have the squared modulus integrable with weight I. If we make the prolongation of these functions by changing $\mathcal{D}$ in $\mathcal{S} = \{(k, u, \varphi)|k \in [0, \infty), u \in (-\infty, +\infty), \varphi \in [0, 2\pi)\}$, and setting $f(k, u, \varphi) = 0$ if $(k, u, \varphi) \notin \mathcal{D}$, then we obtain a subspace of $L^2_I(\mathcal{S})$ (which represents the Hilbert space of functions defined in $\mathcal{S}$ and with the squared modulus integrable with weight I). We note this subspace by $L^2_I(\mathcal{D})$ and notice that the transformation from $L^2(\mathbb{R}^3)$ to $L^2_I(\mathcal{D})$ is a unitary one.

3.2. The determination of the eigenvalues and eigenfunctions of the toroidal momentum operator

In this subsection we determine a complete set of orthogonal functions, formed by eigenfunctions of the toroidal momentum operator.

As we stated at the beginning of this section (see (4)), we can find a common set of eigenfunctions for both the $\hat{T}_3$ projection of the toroidal momentum operator and the $\hat{L}_3$ projection of the angular momentum operator. Because of the very simple form of the $\hat{T}_3$ operator in the (k, u, φ) variables, we can find eigenfunctions of the form

$$\mathcal{T}(k, u, \varphi) = \mathcal{K}(k)\mathcal{M}(\varphi)\mathcal{T}_3(u)$$

where $\mathcal{T}_3$ is an eigenfunction of the T_3 operator, and depends only on the u variable, $\mathcal{M}$ is an eigenfunction of the L_3 operator (which depends on the φ variable), and $\mathcal{K}$ is a function which depends on the k variable and is still undetermined.

We already know that $\mathcal{M}(\varphi) = \exp(\mathrm{i}m\varphi/\hbar)$. $\mathcal{T}_3$ satisfies the differential equation

$$\hat{T}_3\mathcal{T}_3(u) = \frac{-\mathrm{i}\hbar}{10mc}\frac{\partial}{\partial u}\mathcal{T}_3(u) = t_3\mathcal{T}_3(u)$$

with the evident solution

$$\mathcal{T}_3(u) = \exp\left(\frac{\mathrm{i}}{\hbar}10mct_3u\right).$$

Let us try to impose the normalization condition, denoting by a and b two functions belonging to the complete orthogonal system:

$$\begin{aligned} P &= \int \mathcal{K}_a^*(k)\mathcal{M}_a^*(\varphi)\mathcal{T}_{3a}^*(u)\mathcal{K}_b(k)\mathcal{M}_b(\varphi)\mathcal{T}_{3b}(u)2k^3\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}\varphi \\ &= \delta_{m_a m_b}\delta(k_a - k_b)\delta(t_{3a} - t_{3b}). \end{aligned}$$

After some computation (and according to [9], ch VI.6) we find that K should be of the form

$$\mathcal{K}(k) \sim \frac{1}{k^{3/2}}\delta(k).$$

However, this is a very unsuitable form because we are forced to work only with functions defined on the Oz axis. Thus we impose another condition:

$$\begin{aligned} P &= \delta_{m_a m_b}\int \mathcal{K}_a^*(k)\mathcal{K}_b(k)2k^3\left\{\int_{-a(k)}^{a(k)}\exp\left[-\frac{\mathrm{i}}{\hbar}10mc(t_{3a} - t_{3b})u\right]\mathrm{d}u\right\}\mathrm{d}k \\ &= \delta_{m_a m_b}\delta(k_a - k_b)\delta_{t_{3a}t_{3b}}. \end{aligned}$$

This condition imposes for $\mathcal{K}$ the expression

$$\mathcal{K}_{k_0} \sim \frac{1}{k}\delta(k - k_0)$$

and quantifies the t_3 values at fixed k_0. These values are obtainable from the condition

$$\int_{-a(k)}^{a(k)}\exp\left[-\frac{\mathrm{i}}{\hbar}10mc(t_{3a} - t_{3b})u\right]\mathrm{d}u \sim \delta_{t_{3a}t_{3b}}$$

which gives

$$t_{3k_0} = \frac{\hbar}{10mc}\frac{k_0}{C_\mathrm{a}}N\pi + \text{constant}$$

where the constant is arbitrary, common to all the functions $\mathcal{T}_3(u)$ in a set, for each k_0. This is the case of a hypermaximal operator.

The normalization constant is very simple to determine:

$$K_{\mathrm{n}} = \frac{1}{2\sqrt{C_{\mathrm{a}}}}.$$

So we have determined a complete orthogonal set of functions which span the Hilbert space $L^2(\mathbb{R}^3)$. These functions are also normalized:

$$\begin{aligned}\mathcal{T}(k,u,\varphi) &\equiv \mathcal{K}_{k_0}(k)\mathcal{M}_m(\varphi)\mathcal{T}_{3t_3k_0}(u)\\ &= \frac{1}{2\sqrt{C_{\mathrm{a}}}}\frac{1}{k}\delta(k-k_0)\exp\left(\frac{\mathrm{i}}{\hbar}m\varphi\right)\exp\left(\frac{\mathrm{i}}{\hbar}10mct_{3k_0}u\right).\end{aligned}$$

The eigenvalues of the toroidal momentum operator forms a continuous spectrum from $-\infty$ to $+\infty$.

The δ Dirac function appearing in the expression above is imposed by the requirement that the set should be complete and orthogonal in $L^2(\mathbb{R}^3)$. Otherwise the eigenfunctions could have the general form $\mathcal{T}(k,u,\varphi) = f(k,\varphi)\exp(10mct_3u\mathrm{i}/\hbar)$. Other forms for the complete set of eigenfunctions could be obtained if we are working in some specific subspaces of $L^2(\mathbb{R}^3)$, for example, subspaces of functions defined in some domain $D_{\mathrm{u}} \subset \mathbb{R}^3$, with the property that its analogue in the space of (k,u,φ) variables is bounded from above by a fixed value of 'u' and from below by '$-u$', but we do not intend to analyse various special cases here. Examples of curves $u =$ constant are plotted in figure 4.

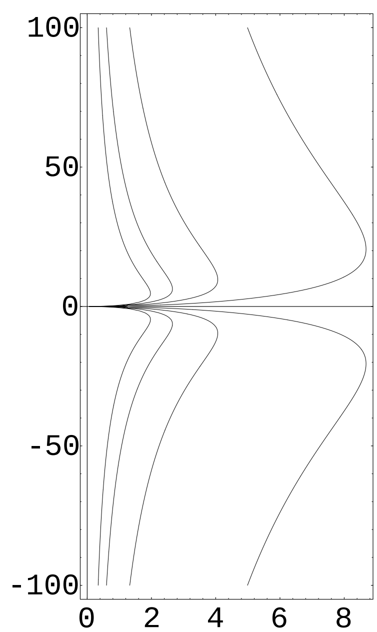

Figure 4. Curves $u(\rho, z) =$ constant, in a $\varphi =$ constant plane.

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